Chern numbers of Chern submanifolds

K. E. Feldman*

Department of Mathematics and Statistics, University of Edinburgh,
James Clerk Maxwell Building, Mayfield Road, Edinburgh, Scotland,
EH9 3JZ

Abstract

We present a solution of the generalized Hirzebruch problem on the relations between the Chern numbers of a stably almost complex manifold and the Chern numbers of its virtual Chern submanifolds.

1 Introduction

A cobordism class $\alpha \in \Omega^k(M)$ of a smooth manifold M can be realized by a smooth submanifold $L \subset M \times \mathbb{R}^N$ of codimension k for N large enough and L is unique up to bordism. We call such a submanifold a virtual submanifold of M. For example, the virtual submanifold corresponding to the Euler class $\chi(\xi)$ of a vector bundle ξ is realized by the zero set of any generic section $s : M \to \xi$. If the cobordism theory $\Omega^*(\cdot)$ has a special structure then the normal bundle of the virtual submanifold L also has the corresponding structure. It is natural to expect that virtual submanifolds inherit some properties from the original manifold. In [5] the authors found a non-trivial relation between the signatures of an almost complex manifold M^{4n} and the virtual submanifolds $[P_k(\tau(M^{4n}))]$ dual to the complex cobordism Pontrjagin classes $P_k(\tau) \in U^{4k}(M^{4n})$ [7]. They posed a more general question: What are all the divisibility relations between the Chern numbers of a stably almost complex manifold and its virtual tangent Chern submanifolds? This is a generalization of the classical Hirzebruch problem [9] on the divisibility restrictions for the Chern numbers of a stably almost complex manifold, which was solved by Hattori and Stong [8, 15].

In the present paper we find all the relations between the Chern numbers of a stably almost complex manifold M^{2n} and its virtual submanifolds $[c_k(\eta)]$ defined by the Chern classes $c_k(\eta) \in U^{2k}(M^{2n})$ of a complex vector bundle η. The result of [5] is a particular case of our general theorem.

For the solution we construct a multiplicative transformation τ of complex cobordism theory $U^*(\cdot)$ to ordinary cohomology theory $H^*(\cdot, \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q})$, where following [12]

*e-mail:feldman@maths.ed.ac.uk

$\Lambda[y], \Lambda[z]$ are the rings of symmetric polynomials over $\mathbb{Z}$ in variables y_i and z_i, $i = 1, 2, ...,$ respectively. The key tools for the construction are the Chern–Dold character [6] and the Landweber–Novikov operations [11, 13]. Applying the Riemann–Roch theorem [1] to the transformation τ we can express any Chern number of the virtual Chern submanifold in terms of the Chern numbers of the original manifold. Together with the Hattori–Stong theorem this expression describes all the relations between Chern numbers.

The paper is organized in the following way. In Section 2 we define the transformation $\tau : U^*(\cdot) \to H^*(\cdot; \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q})$ and calculate its Todd genus.

In Section 3 we apply the Riemann–Roch theorem to the transformation τ and obtain an expression for the normal Chern numbers of the normal virtual Chern submanifolds $[c_k(\nu(M^{2n}))]$ in terms of the Chern numbers of the original manifold M^{2n}.

In Section 4 we present some examples of numerical relations between Chern numbers of Chern submanifolds. In particular, we deduce the formula obtained by Buchstaber and Veselov in [5].

In Section 5 we prove a general theorem on the relations between Chern numbers of virtual Chern submanifolds $[c_k(\eta)]$ corresponding to a complex vector bundle η over a stably almost complex manifold M. Using the Hattori–Stong theorem we deduce a new divisibility condition for the cohomology Chern classes. As an example we prove that the Euler characteristic of an almost complex manifold whose tangent bundle possesses a complex line subbundle is even.

2 The transformation τ and its Todd class

Denote the Landweber–Novikov operation corresponding to the symmetric polynomial $f \in \Lambda[z]$ by $\mathbf{f}$. For the Thom class $u_m \in U^{2m}(MU(m))$ and the tautological bundle η_m over $BU(m)$ we have

$$\mathbf{f}(u_m) = u_m f(\eta_m),$$

where $f(\eta_m)$ is the Chern class in complex cobordism corresponding to the symmetric polynomial f. Let $p_k(z) \in \Lambda[z]$ be the k-th power sum of z_i, $i = 1, 2, ...$ Denote the monomial symmetric function corresponding to the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq 0)$ by $m_\lambda(z)$. One defines the complete symmetric functions $h_\lambda(z)$ using the identity [12]

$$\prod_{i,j=1}^{\infty} (1 - z_i y_j)^{-1} = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(z) p_k(y)}{k}\right) = 1 + \sum_{\{\lambda \mid |\lambda| > 0\}} h_\lambda(z) m_\lambda(y). \tag{1}$$

We combine all the Landweber–Novikov operations into one multiplicative operation $\mathcal{S}_{(z)}$:

$$\mathcal{S}_{(z)} = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(z) \mathbf{p_k}}{k}\right) = \sum_{\lambda} h_\lambda(z) \mathbf{m}_\lambda.$$

The operation $\mathcal{S}_{(z)}$ acts as a ring endomorphism on $U^*(X) \otimes \Lambda[z]$ for any topological space X. Moreover, this operation is invertible (see [4] for further algebraic properties of $\mathcal{S}_{(z)}$).

In [6] the Chern–Dold character $ch_U : U^*(X) \to H^*(X, \Omega_*^U \otimes \mathbb{Q})$ was constructed which is the canonical embedding for $X = \{pt\}$. The value of ch_U on the Thom class $u \in U^2(MU(1))$ is given by the formula [6]:

$$ch_U(u) = x + \sum_{n=1}^{\infty} [N^{2n}] \frac{x^{n+1}}{(n+1)!},$$

where x is the Thom class of $MU(1)$ in ordinary cohomology, N^{2n} is a closed stably almost complex manifold defined uniquely by the conditions $m_\lambda(N^{2n}) = 0$ for $\lambda \neq (n)$ and $m_{(n)}(N^{2n}) = (n+1)!$ (we denote the normal Chern number of a stably almost complex manifold M^{2n} corresponding to the normal Chern class $f(\nu(M^{2n}))$, $\deg f = n$, by $f(M^{2n})$).

Define the "universal" symmetric genus

$$S^*_{(y)} : \Omega_*^U \to \Lambda[y] \qquad \text{by} \qquad S^*_{(y)}([M^{2n}]) = \sum_{\{\lambda \| \lambda| = n\}} h_\lambda(y) m_\lambda(M^{2n}).$$

From (1) it follows that the generating function of the universal symmetric genus $S^*_{(y)}$ is

$$S^*_{(y)}(x) = \prod_{i=1}^{\infty} (1 - y_i x)^{-1}.$$

The transformation

$$H^*(X, \Omega_*^U \otimes \Lambda[z] \otimes \mathbb{Q}) \to H^*(X, \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q})$$

determined through the coefficient ring homomorphism $S^*_{(y)} \otimes id \otimes id : \Omega_*^U \otimes \Lambda[z] \otimes \mathbb{Q} \to \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q}$ is also denoted by $S^*_{(y)}$.

Let τ be the composition $S^*_{(y)} \circ ch_U \circ \mathcal{S}_{(z)}$. Obviously, τ is a multiplicative transformation of cohomology theories

$$\tau : U^*(X) \to H^*(X, \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q}).$$

Recall that for any multiplicative transformation $\mu : h_1^*(\cdot) \to h_2^*(\cdot)$ of complex oriented cohomology theories $h_1^*(\cdot), h_2^*(\cdot)$ the Todd class T_μ is defined via the Riemann–Roch theorem [1]:

$$p_!^{h_2}(\mu(x) \cdot T_\mu(\nu)) = \mu\left(p_!^{h_1}(x)\right),$$

where $p : M_1 \to M_2$ is a map of smooth manifolds with the stably complex oriented normal bundle $\nu \cong p^*(\tau(M_2)) - \tau(M_1)$, $p_!^{h_1}$ and $p_!^{h_2}$ are the corresponding Gysin maps in $h_1^*(\cdot)$ and $h_2^*(\cdot)$ respectively, $x \in h_1^*(M_1)$.

Theorem 2.1 *The Todd class corresponding to τ is*

$$T_\tau(\eta) = \prod_{i=1}^{n} \left(\prod_{j=1}^{\infty} (1 - y_j x_i)^{-1} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z_k x_i}{\prod_{j=1}^{\infty} (1 - y_j x_i)} \right)^{-1} \right),$$

here $n = \dim_{\mathbb{C}} \eta$, x_i, $i = 1, \ldots, n$, are the Chern roots in integral cohomology of the complex vector bundle η.

Proof. By splitting principle it is sufficient to verify this formula for the case $n = 1$. For the Thom class $u \in U^2(MU(1))$ we obtain

$$\mathcal{S}_{(z)}(u) = u \cdot \exp\left(\sum_{k=1}^{\infty} \frac{p_k(z)u^k}{k}\right) = u \prod_{k=1}^{\infty}(1 - z_k u)^{-1}; \tag{2}$$

For the manifolds N^{2n} in the definition of the Chern-Dold character one has $S^*_{(y)}(N^{2n}) = (n+1)!h_{(n)}(y)$. Then (see also [12])

$$S^*_{(y)} \circ ch_U(u) = x + \sum_{n=1}^{\infty} h_{(n)}(y)x^{n+1} = x \prod_{j=1}^{\infty}(1 - y_j x)^{-1}. \tag{3}$$

The statement in the theorem now follows using the multiplicative property of $S^*_{(y)} \circ ch_U$.

Remark 2.1 *The multiplicative operation $\mathcal{S}_{(z)}$ transforms a stably almost complex manifold into the collection of all Chern submanifolds. The Chern–Dold character maps them into the coefficient ring Ω^U_*. The operation $S^*_{(y)}$ calculates the Chern numbers of the elements of Ω^U_*.*

3 Riemann–Roch theorem for τ

Consider a smooth compact closed connected manifold M^{2n} without boundary with fixed almost complex structure on its stable normal bundle $\nu(M^{2n})$. Every Chern class of the normal bundle $\nu(M^{2n})$ can be represented by a submanifold of $M^{2n} \times \mathbb{R}^N$ with almost complex normal structure for N large enough. We will denote the manifold corresponding to the characteristic class $f(\nu(M^{2n})) \in U^*(M^{2n})$ by $[f(\nu(M^{2n}))]$. Let $\langle x, [M^{2n}]\rangle$ be the Kronecker product of $x \in H^{2n}(M^{2n}, \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q})$ with the fundamental class $[M^{2n}] \in H_{2n}(M^{2n}, \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q})$; if $\deg x \neq 2n$ we put $\langle x, [M^{2n}]\rangle = 0$.

Theorem 3.1 *The following equality holds:*

$$\langle T_\tau(\nu(M^{2n})), [M^{2n}]\rangle = S^*_{(y)}(M^{2n}) + \sum_{\{\lambda||\lambda|>0\}} h_\lambda(z) S^*_{(y)}([m_\lambda(\nu(M^{2n}))]). \tag{4}$$

Remark 3.1 *The left side of this equality depends only on the Chern numbers of the manifold M^{2n}. The right side depends on the Chern numbers of the virtual Chern submanifolds. Moreover, because of the linear independence of $h_\lambda(z)$ in $\Lambda[z]$ and $m_\lambda(y)$ in $\Lambda[y]$, the Chern numbers of a Chern submanifold can be expressed in terms of the Chern numbers of M^{2n} using (4).*

Proof. Denote the constant map $M^{2n} \to \{pt\}$ by p. The corresponding Gysin map in complex cobordism is denoted by $p^U_!$ and that in cohomology by $p^H_!$. There is a canonical cobordism class $\sigma \in U^0(M^{2n})$ defined by the identity map $M^{2n} \to M^{2n}$. It is obvious that

$$p^U_!(\sigma) = [M^{2n}, \nu(M^{2n})] \in \Omega_U^{-2n}(pt) = \Omega^U_{2n}. \tag{5}$$

We apply the Riemann–Roch theorem in the form given in [1] to the transformation τ and the element σ. Using (5) we obtain

$$p_!^H(\tau(\sigma)\cdot T_\tau(\nu(M^{2n}))) = \tau([M^{2n},\nu(M^{2n})]).$$

Because the normal bundle of the identity map $M^{2n}\to M^{2n}$ is trivial, we have $\tau(\sigma) = S^*_{(y)}\circ ch_U(\sigma)$. According to the results of [6],

$$(S^*_{(y)}\circ ch_U)(\sigma) = 1.$$

Thus,

$$p_!^H(\tau(\sigma)\cdot T_\tau(\nu(M^{2n}))) = \langle T_\tau(\nu(M^{2n})),[M^{2n}]\rangle.$$

On the other hand, the normal bundle of the constant map $M^{2n}\to\{pt\}$ is $\nu(M^{2n})$. Thus,

$$\mathcal{S}_{(z)}([M^{2n},\nu(M^{2n})]) = [M^{2n}] + \sum_{\{\lambda||\lambda|>0\}} h_\lambda(z)[m_\lambda(\nu(M^{2n}))]\in\Omega^U_*.$$

Note that ch_U is the identity on the coefficient ring Ω^U_*. Applying $S^*_{(y)}$ we obtain Equation (4).

Remark 3.2 *From Theorem 3.1 it follows that the Chern numbers of any virtual Chern submanifold of a stably almost complex manifold depend only on the cobordism class of the original manifold. This is to be expected because the composition of any two Landweber–Novikov operations is again a Landweber–Novikov operation.*

4 Examples

Various substitutions into (4) lead us to families of relations between Chern numbers of stably almost complex manifold and its Chern submanifolds. Let us consider the Hirzebruch genus χ_y [10] which arises from the power series

$$Q(x) = \frac{x(1+y\mathrm{e}^{-x(1+y)})}{1-\mathrm{e}^{-x(1+y)}}.$$

Theorem 4.1 *For the tangent virtual Chern submanifolds $[c_k(\tau(M^{2n}))]$ the following relation holds*

$$(1+y)^n T(M^{2n}) = \chi_y(M^{2n}) + \sum_{k=1}^{n} y^k\chi_y([c_k(\tau(M^{2n}))]), \qquad (6)$$

where $T(M^{2n})$ is the classical Todd genus of the manifold M^{2n}.

Note: the Chern numbers used in this theorem are those of the tangent bundle.

Proof. Substitute $z_1=-y$, $z_2=z_3=\ldots=0$ into (4) and choose the values of y_k so that

$$S^*_{(y)}(x) = \prod_{j=1}^{\infty}(1-y_jx)^{-1} = Q(x)^{-1}.$$

If we calculate the normal Chern number corresponding to the $\frac{1}{Q}$-genus of any stably almost complex manifold, we obtain its χ_y genus. Because $c_k(\tau(M^{2n})) = (-1)^k h_k(\nu(M^{2n}))$, for this choice of variables the right hand side of (4) becomes

$$\chi_y(M^{2n}) + \sum_{k=1}^{n} y^k \chi_y([c_k(\tau(M^{2n}))]).$$

Now we calculate the left hand side of (4) for this choice of variables. We have the following equality

$$Q(x)^{-1}\left(1+\frac{yx}{Q(x)}\right)^{-1} = (Q(x)+yx)^{-1} = \frac{1-\mathrm{e}^{-x(1+y)}}{x(1+y)}.$$

The corresponding normal Chern number of M^{2n} is exactly the classical Todd genus of M^{2n} multiplied by $(1+y)^n$. Thus, Equation (6) follows.

Remark 4.1 *Putting $y = -1$ or 1 into (6) we obtain equations on the Euler characteristics and the signatures of the tangent virtual Chern submanifolds.*

In the same manner we can obtain relations between the Chern numbers of a stably almost complex manifold and its virtual Pontrjagin submanifolds. Recall that the tangent virtual Pontrjagin submanifold $[P_k(\tau(M^{4n}))]$ is defined as $[(-1)^k c_{2k}(\mathbb{C}\otimes\tau(M^{4k}))]$.

Theorem 4.2 *For the signatures of the submanifolds $[P_k(\tau(M^{4n}))]$ of a stably almost complex manifold M^{4n} the following relation holds*

$$2^{4n}\hat{A}(M^{4n}) = \sigma(M^{4n}) + \sum_{k=1}^{n}(-1)^k\sigma([P_k(\tau(M^{4n}))]), \tag{7}$$

where $\hat{A}(M)$ is the $\hat{A}$-genus of M.

Proof. Substitute $z_1 = -z_2 = 1$, $z_3 = z_4 = ... = 0$ into (4) and choose the values of y_k so that

$$S^*_{(y)}(x) = \prod_{j=1}^{\infty}(1-y_jx)^{-1} = \frac{tanh(x)}{x} = \tilde{L}(x).$$

If we calculate the normal Chern number corresponding to the $\tilde{L}$-genus of any stably almost complex manifold, we obtain its signature. For a stably almost complex manifold M^{4n} the Pontrjagin classes can be expressed in terms of the Chern roots r_j, $j = 1, ..., 2n$, of the tangent bundle:

$$\sum_{k=0}^{n} P_k t^{2k} = \prod_{j=1}^{2n}(1+ir_jt)(1-ir_jt),$$

where t is a formal parameter. Using again the fact $c_k(\tau(M^{4n})) = (-1)^k h_k(\nu(M^{4n}))$, for this choice of variables, the right hand side of (4) becomes

$$\sigma(M^{4n}) + \sum_{k=1}^{n}(-1)^k\sigma([P_k(\tau(M^{4n}))]).$$

Now we calculate the left hand side of (4) for this case. We obtain the following equality

$$\tilde{L}(x)\left(1-x^2\tilde{L}(x)^2\right)^{-1}=\left(\tilde{L}(x)^{-1}-x^2\tilde{L}(x)\right)^{-1}=\frac{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}{4x}=\hat{A}(4x)^{-1}.$$

The corresponding normal Chern number of M^{4n} is exactly the $\hat{A}$-genus of M^{4n} multiplied by 2^{4n}. Thus, Equation (7) follows.

Remark 4.2 *Denote the number of ones in the binary expansion of the number n by $\alpha(n)$. It is well known that the $\hat{A}$-genus of a stably almost complex manifold M^{4n} multiplied by $2^{4n-\alpha(n)}$ is an integer [2]. Thus, from Theorem 4.2 we can deduce the main result of [5]*

$$\sigma(M^{4n})+\sum_{k=1}^{n}(-1)^k\sigma([P_k(\tau(M^{4n}))])=0(\mathrm{mod}2^{\alpha(n)}).$$

5 Chern numbers of general Chern submanifolds

Virtual Chern submanifolds $[m_\lambda(\eta)]$ can be defined for any stably almost complex bundle η over a stably almost complex manifold M with the fixed complex structure in the normal bundle $\nu=\nu(M)$. It turns out that the Chern numbers of such submanifolds can also be expressed in terms of the Chern numbers of M. Introduce two power series with coefficients in $\Lambda[y]\otimes\Lambda[z]$ and $\Lambda[y]$ respectively:

$$T_1(x)=\prod_{k=1}^{\infty}\left(1-\frac{z_kx}{\prod_{j=1}^{\infty}(1-y_jx)}\right)^{-1},\qquad T_2(x)=\prod_{i=1}^{\infty}(1-y_ix)^{-1}.$$

For a complex vector bundle η, $\dim_{\mathbb{C}}\eta=n$, over a base space X we define two non-homogeneous elements in $H^*(X,\Lambda[y]\otimes\Lambda[z])$ by the formulae

$$T_1(\eta)=\prod_{i=1}^{n}T_1(x_i),\qquad T_2(\eta)=\prod_{i=1}^{n}T_2(x_i),$$

where x_i are the Chern roots of η in ordinary cohomology.

Theorem 5.1 *The Chern numbers of the Chern submanifolds $[m_\lambda(\eta)]$ of a stably almost complex manifold M are related to the Chern numbers of M through the formula*

$$\langle T_1(\eta)\cdot T_2(\nu),[M]\rangle=S^*_{(y)}(M)+\sum_{\{\lambda||\lambda|>0\}}h_\lambda(z)S^*_{(y)}([m_\lambda(\eta)]).\tag{8}$$

Proof. For any stably almost complex vector bundle η over a base space M define a power series in $\Lambda[z]\otimes U^*(M)$:

$$\mathcal{M}(\eta)=1+\sum_{\{\lambda||\lambda|>0\}}h_\lambda(z)m_\lambda(\eta).$$

Applying the Riemann-Roch theorem to the constant map $p : M \to \{pt\}$, the element $\mathcal{M}(\eta) \in \Lambda[z] \otimes U^*(M)$ and the transformation

$$S^*_{(y)} \circ ch_U : U^*(\cdot) \otimes \Lambda[z] \to H^*(\cdot, \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q}),$$

we obtain

$$p_!^H \left(\left(S^*_{(y)} \circ ch_U(\mathcal{M}(\eta))\right) \cdot T(\nu(M)) \right) = S^*_{(y)} \circ ch_U \left(p_!^U \mathcal{M}(\eta)\right), \tag{9}$$

where $T(\xi)$ is the Todd genus of the transformation $S^*_{(y)} \circ ch_U$. It is obvious that the right hand side of (9) coincides with the right hand side of (8). It is also a simple corollary of (3) that $T(\xi) = T_2(\xi)$. By the splitting principle to calculate $S^*_{(y)} \circ ch_U(\mathcal{M}(\eta))$ it is sufficient to deal with only one-dimensional vector bundles, i.e. with the tautological bundle over $CP(\infty)$. In this case

$$\mathcal{M}(\eta_1) = \prod_{i=1}^{\infty} (1 - z_i u)^{-1},$$

where u is the canonical generator of $U^*(CP(\infty))$. Applying (3) we obtain that the right hand side of (9) equals $\langle T_1(\eta) \cdot T_2(\nu), [M] \rangle$.

Various divisibility conditions, which are analogues of those [3] coming from Atiyah–Singer theorem, can be extracted from Theorem 5.1. Consider the genera

$$T_A(\eta) = \prod_{i=1}^{m} \prod_{k=1}^{\infty} \left(1 - \frac{z_k(1 - \mathrm{e}^{-x_i})}{\prod_{j=1}^{\infty} (1 - y_j(1 - \mathrm{e}^{-x_i}))} \right)^{-1},$$

$$T_B(\eta) = \prod_{i=1}^{m} \left(\frac{1 - \mathrm{e}^{-x_i}}{x_i} \prod_{j=1}^{\infty} \left(1 - y_j(1 - \mathrm{e}^{-x_i})\right)^{-1} \right)$$

where x_i, $i = 1, ...m$, are the Chern roots in integral cohomology of a complex vector bundle η, $\dim_{\mathbb{C}} \eta = m$.

Corollary 5.1 *For a complex vector bundle η over a stably almost complex manifold M^{2n} with the normal bundle ν, the Chern class $T_A(\eta)T_B(\nu) \in H^*(M^{2n}, \Lambda[y] \otimes \Lambda[z] \otimes \mathbb{Q})$ evaluated on the fundamental cycle $[M^{2n}]$ lies in $\Lambda[y] \otimes \Lambda[z]$.*

Proof. Consider variables $\tilde{y}_j = \tilde{y}_j(y)$, $j = 1, 2, ...$, such that

$$S^*_{(\tilde{y})}(x) = \prod_{j=1}^{\infty} (1 - \tilde{y}_j x)^{-1} = \frac{1 - \mathrm{e}^{-x}}{x} \prod_{j=1}^{\infty} \left(1 - y_j(1 - \mathrm{e}^{-x})\right)^{-1}.$$

Then for any stably almost complex manifold L we have that $S^*_{(\tilde{y})}(L)$ is a combination of K-theory characteristic numbers with coefficients in $\Lambda[y]$ [14]. According to the Hattori–Stong theorem [8, 15] every K-theory characteristic number is integral. Thus, the corollary follows from Theorem 5.1 after making the substitution $y \to \tilde{y}$.

Corollary 5.2 *Let M^{2n} be an almost complex manifold whose tangent space contains a complex line subbundle. Then the Euler characteristic of M^{2n} is even.*

Proof. Denote the normal bundle of M^{2n} by ν. Let $\tau \cong \tau_1 \oplus \eta_1$ be the splitting of the tangent bundle $\tau \cong \tau(M^{2n})$, where η_1 is a complex line bundle. From Corollary 5.1 we deduce that the power series $\langle T_A(\eta_1 \oplus \nu) T_B(\nu), [M^{2n}] \rangle$ has integral coefficients. Denote the Chern roots in ordinary cohomology of τ_1 by $x_1, x_2, ..., x_{n-1}$, and the first Chern class in ordinary cohomology of η_1 by x. Substitute $z_1 = z$, $z_2 = z_3 = ... = 0$ and $y_1 = y_2 = ... = 0$ into $T_A(\tau_1)$ and $T_B(\nu)$. Because $T_A(\eta_1 \oplus \nu) = T_A^{-1}(\tau_1)$, $T_B(\nu) = T_B^{-1}(\tau_1) T_B^{-1}(\eta_1)$ we obtain, that the coefficient of z^{n-1} in the polynomial

$$\left\langle \left(\frac{x}{1 - \mathrm{e}^{-x}} \prod_{i=1}^{n-1} \left(\frac{x_i (1 - z(1 - \mathrm{e}^{-x_i}))}{1 - \mathrm{e}^{-x_i}} \right) \right), [M^{2n}] \right\rangle$$

is an integer. One can easily calculate that this coefficient is

$$\left\langle \frac{x}{2} \prod_{i=1}^{n-1} x_i, [M^{2n}] \right\rangle = \frac{\chi(M^{2n})}{2},$$

where $\chi(M^{2n})$ is the Euler characteristic of M^{2n}. Thus, $\chi(M^{2n})$ is even.

Conclusion

We would like to point out that there is a real version of most of the material of the paper. To reproduce the results above in oriented cobordism theory one notes that the forgetful map $\Omega_*^U \to \Omega_*^{SO}/\mathrm{Tors}$ is an epimorphism. There is also the version of the paper for Stiefel–Whitney classes. It will be interesting to understand what both the versions mean in terms of triangulated manifolds.

The final remark concerning Theorem 3.1 is that it gives a complete description of the Landweber–Novikov algebra action on the cobordism ring of a point. This might be helpful for calculations of homotopy groups of spheres.

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References

[1] M. F. Atiyah, F. Hirzebruch, *Riemann–Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. 65 (1959), 276–281.

[2] M. F. Atiyah, F. Hirzebruch, *Quelques theoremes de non-plongement pour les varietes differentiables*, Bull. Soc. Math. France 87 (1959), 383–396.

[3] M. F. Atiyah, *Vector fields on manifolds*, Arbeitsgemeinschaft fur Forschung des Landes Nordrhein-Westfalen, Heft 200 Westdeutscher Verlag, Cologne 1970.

[4] B. I. Botvinnik, V. M. Buchstaber, S. P. Novikov, S. A. Yuzvinsky, *Algebraic aspects of multiplication theory in complex cobordisms* (in Russian), Uspekhi Mat. Nauk 55(2000), no. 4, 5–24.

[5] V. M. Buchstaber, A. P. Veselov, *On a remarkable functional equation in the theory of generalized Dunkl operators and transformations of elliptic genera*, Math. Zeitschr. 223 (1996), 595–607.

[6] V. M. Buchstaber, *The Chern–Dold character in cobordisms* (in Russian), Mat. Sbornik 83(125) (1970), 575–595.

[7] P. E. Conner, E. E. Floyd, *The relation of cobordism to K-theories*, Lecture Notes in Math. 28(1966).

[8] A. Hattori, *Integral characteristic numbers for weakly almost complex manifolds*, Topology 5 (1966), 259–280.

[9] F. Hirzebruch, *Some problems on differentiable and complex manifolds*, Ann. of Math. (2) 60 (1954), 213–236.

[10] F. Hirzebruch, T. Berger, R. Jung, *Manifolds and modular forms*, A publication of the Max–Planck–Institute fur Mathematik, Bonn, 1992.

[11] P. S. Landweber, *Cobordism operations and Hopf algebras*, Trans. Amer. Math. Soc. 129 (1967), 94–110.

[12] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford mathematical monographs, 1995.

[13] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory* (in Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 31(1967), 855–951.

[14] L. Smith, R. E. Stong, *Dirichlet series and homology theories*, in P. S. Landweber (Ed.), *Elliptic curves and modular forms in algebraic topology*, Lecture Notes in Math. 1326, pp.134–149 (1988).

[15] R. E. Stong, *Relations among characteristic numbers I*, Topology 4 (1965), 267–281.